# Parametric Analysis of Multiple Interconnects via Canonical Reduced Order Modeling*

Zhigang Hao and Guoyong Shi
School of Microelectronics, Shanghai Jiao Tong University, Shanghai 200030, China
{haozhigang, shiguoyong}@ic.sjtu.edu.cn

## ABSTRACT

For design nodes at 65nm and below, timing will essentially be a statistical measure of the fabricated circuit and heavily correlated with process variation. This paper proposes a novel parametric interconnect analysis using canonical reduced order modeling. Models in canonical forms have the feature of a small number of free model parameters. This property can be made use of effectively for parametric analysis via interpolation. Experimental results demonstrate the effectiveness of the proposed methodology.

## 1. INTRODUCTION

State-of-the-art integrated circuit design has entered the nanometer design era. The process variation problem is one of the most serious challenges faced by the IC designers today [1]. Timing has to be analyzed in the statistical framework [2]. Current research is being directed toward statistical analysis of interconnects.

Traditional model order reduction techniques have found successful applications to interconnect modeling and efficient timing analysis. Extension of the classical methods to a statistical formulation is not trivial. Monte Carlo based repeated model reduction for collecting the statistics is doomed to be too costly. A number of methods for better efficiency have been proposed in the literature. The two-stage interval analysis technique [3] is probably the most promising one proposed so far.

Model order reduction (MOR) is still considered a powerful toolkit yet to be explored for statistical interconnect modeling and analysis. Exploring work along this line includes a combination of MOR with interval analysis [3], symbolic approaches [4], and a combination of the Padé-via-Lanczos (PVL)[7] algorithm with interpolation [5].

This paper attempts to investigate a new approach to compact reduced order modeling and explore its potentials for multi-input multi-output (MIMO) interconnect modeling and parametric analysis. It is observed that a reduced-order model in the observability canonical form [6] reveals special compactness; namely, only a small number of matrix entries (called *model data*) determine the reduced order model. This property is particularly useful for parametric/statistical interconnect modeling and timing analysis.

## 2. MODELING IN OBSERVABILITY CANONICAL FORM

General dynamic linear systems can be represented in canonical forms in which the system models exhibit simplistic regular forms that make system realization easy [6]. One of the canonical forms is called the *observability canonical form.*

Consider the following single-input-single-output (SISO) system

$$\begin{cases} \dot{x} &= Ax + bu, \qquad x \in \mathbb{R}^n \\ y &= c^{\mathrm{T}} x \end{cases} \tag{1}$$

where $b$ and $c$ are two column vectors, $c^{\mathrm{T}}$ is the transposed vector of $c$. Variables $u$ and $y$ are scalar input and output signals, respectively. Let $\lambda(s)$ be the characteristic polynomial of matrix $A$, i.e.

$$\lambda(s) = |sI - A| = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0. \tag{2}$$

By the *Cayley-Hamilton Theorem* the following matrix equation holds [6]

$$A^n = -a_0 I - a_1 A - \cdots - a_{n-1} A^{n-1}. \tag{3}$$

In linear system theory, the matrix pair ($c^{\mathrm{T}}$, $A$) is called *observable* if the following matrix is nonsingular

$$\begin{bmatrix} c^{\mathrm{T}} \\ c^{\mathrm{T}} A \\ \vdots \\ c^{\mathrm{T}} A^{n-1} \end{bmatrix} \triangleq T^{-1}.$$

Using the columns of matrix $T$ as the coordinate basis, the original system model represented in this new coordinate system takes the following *observability canonical form*:

$$\begin{aligned} \hat{A} &= T^{-1}AT = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, \\ \hat{b} &= T^{-1}b = \begin{bmatrix} c^{\mathrm{T}} b \\ c^{\mathrm{T}} A b \\ \vdots \\ c^{\mathrm{T}} A^{n-1} b \end{bmatrix} \triangleq \begin{bmatrix} m_0 \\ m_1 \\ \vdots \\ m_{n-1} \end{bmatrix}, \\ \hat{c}^{\mathrm{T}} &= c^{\mathrm{T}} T = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}, \end{aligned}$$

where the $a_i$'s are the coefficients of the characteristic polynomial of matrix $A$. The entries $m_i = c^{\mathrm{T}} A^i b = \hat{c}^{\mathrm{T}} \hat{A}^i \hat{b}$ $(i = 0, 1, \cdots)$ are called *Markov coefficients* or *moments*.

*Remark 1:* It is important to note that the above model in the observability canonical form is determined by only $2n$ model data, $a_i$ and $m_i$ for $i = 0, \cdots, n-1$. We shall take this advantage for parametric reduced-order modeling.

The above formulation for SISO models can easily be extended to multi-input multi-output (MIMO) models. Many linearized circuit models take the following form (with $r$ inputs and $p$ outputs):

$$\left\{ \begin{array}{rcll} C\dot{x} &=& -Gx + Fu, & x \in \mathbb{R}^n;\ \ u \in \mathbb{R}^r \\ y &=& Lx, & y \in \mathbb{R}^p \end{array} \right. \tag{4}$$

Sometimes the matrix $C$ would be nonsingular (so called *Differential-Algebraic Equations*, DAE). We may choose a real number $s_0$ so that the matrix $(G + s_0 C)$ is nonsingular. This makes it possible to expand the transfer function of the model (4) in the following manner:

$$\begin{aligned} H_M(s) &= L\left[C(s - s_0) + (G + Cs_0)\right]^{-1} F \\ &= L\left[A(s - s_0) + I\right]^{-1} B \\ &= \sum_{j=0}^{\infty} (-1)^j M_j (s - s_0)^j, \end{aligned}$$

where $M_j = LA^j B$ is the $j$th moment, $A = (G + s_0 C)^{-1} C$, and $B = (G + s_0 C)^{-1} F$. This kind of treatment has been standard in the literature [7]. The real number $s_0$ is called the *expansion point* in the Taylor expansion for moment matching.

The above transfer function $H_M(s+s_0)$ also represents the following linear system

$$\left\{ \begin{array}{rcll} A\dot{x} &=& -x + Bu, & x \in \mathbb{R}^n;\ \ u \in \mathbb{R}^r \\ y &=& Lx, & y \in \mathbb{R}^p. \end{array} \right. \tag{5}$$

The model data $(L, A, B)$ can be transformed to the observability canonical form analogously to the SISO case.

Let $L = \begin{bmatrix} \ell_1 & \ell_2 & \cdots \ell_p \end{bmatrix}^{\mathrm{T}}$ and $B = \begin{bmatrix} b_1 & b_2 & \cdots & b_r \end{bmatrix}$, where $\ell_i$ and $b_j$ are all column vectors. We have

$$\begin{aligned} H_M(s + s_0) &= L(sA + I)^{-1} B \\ &= \begin{bmatrix} \ell_1^{\mathrm{T}} \\ \ell_2^{\mathrm{T}} \\ \vdots \\ \ell_p^{\mathrm{T}} \end{bmatrix} (sA + I)^{-1} \begin{bmatrix} b_1 & b_2 & \cdots & b_r \end{bmatrix} \\ &= \begin{bmatrix} h_{11}(s) & h_{12}(s) & \cdots & h_{1r}(s) \\ h_{21}(s) & h_{22}(s) & \cdots & h_{2r}(s) \\ \vdots & \vdots & \ddots & \vdots \\ h_{p1}(s) & h_{p2}(s) & \cdots & h_{pr}(s) \end{bmatrix} \end{aligned}$$

where $h_{ij}(s) = \ell_i^{\mathrm{T}} (sA + I)^{-1} b_j,\ 1 \le i \le p,\ 1 \le j \le r$. This matrix transfer function consists of $rp$ scalar transfer functions. The procedure presented next can reduce these scalar transfer functions $h_{ij}(s)$ to lower-order ones.

## 3. A REDUCTION ALGORITHM FOR MIMO MODELS

The following algorithm reduces a MIMO model to the observability canonical form.

**Observability Canonical Form Reduction (OCFR) Algorithm**

Step 1. Construct a $q$th order Krylov subspace using the Block Arnoldi Algorithm (to be stated in the next section) using the model data $(A^{\mathrm{T}}, L^{\mathrm{T}})$:

$$\mathcal{K}_q(A^{\mathrm{T}}, L^{\mathrm{T}}) = \mathrm{span}\left[L^{\mathrm{T}}, A^{\mathrm{T}} L^{\mathrm{T}}, \cdots, A^{\mathrm{T}(q-1)} L^{\mathrm{T}}\right]$$

Step 2. Find the coefficient matrices of $LA^q$ expressed in terms of a block linear combination of $LA^i$ $(i = 1, \cdots, q-1)$:

$$LA^q \approx -M_0 L - M_1 LA - \cdots - M_{q-1} LA^{(q-1)},$$

where $M_i$ $(i = 0, \cdots, q-1))$ are $p \times p$ matrices. (The computation of $M_i$'s will be given in the next section.)

Step 3. Form the following model in the block observability canonical form for the given $q$:

$$A_q = \begin{bmatrix} 0 & I_p & & & \\ & 0 & I_p & & \\ & & & \ddots & \\ & & & & I_p \\ -M_0 & -M_1 & -M_2 & \cdots & -M_{q-1} \end{bmatrix},$$

$$B_q = \begin{bmatrix} LB \\ LAB \\ \vdots \\ LA^{q-1}B \end{bmatrix},$$

$$L_q = \begin{bmatrix} I_p & 0 & \cdots & 0 \end{bmatrix}.$$

Step 4. Check the stability and passivity of the obtained model.

It can be shown that the $q$ leading matrix moments of the reduced order model, i.e., $L_q A_q^i B_q$ for $i = 0, \cdots, q-1$, match exactly those of the full-order model.

Let

$$M = \begin{bmatrix} M_0 & \cdots & M_{q-1} \end{bmatrix}$$

and

$$W_q^{\mathrm{T}} = \begin{bmatrix} L \\ LA \\ \vdots \\ LA^{q-1} \end{bmatrix} \in \mathbb{R}^{pq \times n}.$$

One can further show that the error between the reduced-order model and the full-order model is controlled by the norm

$$error \triangleq \left\| LA^q + MW_q^{\mathrm{T}} \right\| .$$

If this error vanishes, that means the reduced-order model represents exactly the full-order model without error.

## 4. NUMERICAL COMPUTATION

In implementation, we usually do not compute the vectors $(A^i)^{\mathrm{T}} L^{\mathrm{T}}$ directly because the power iteration is not numerically stable. Typically the subspace spanned by the columns of $W_q$ is computed by a numerically stable *Block Arnoldi Algorithm*[9], which computes iteratively the orthonormal basis vectors of the Krylov subspace $\mathcal{K}_q(A^{\mathrm{T}}, L^{\mathrm{T}})$.

**Block Arnoldi Algorithm:**

```
In:  A^T, L^T, q.
Out: W~_q, V_{q+1}, H_q, H_{q+1,q}.

(V_1, R_11) = QR(L^T);
for (j = 1; j <= q; j++) {
    U_j = A^T V_j;
    for (i = 1; i <= j; i++) {
        H_{i,j} = V_i^T U_j;
        U_j = U_j - V_i H_{i,j};
    }
    (V_{j+1}, H_{j+1,j}) = QR(U_j);
}
W~_q = [V_1 V_2 ... V_q];
H_q = (H_{i,j}), i,j = 1,...,q;
```

In the algorithm, $QR(\cdot)$ denotes the QR factorization.

Since the columns of $\widetilde{W}_q$ and the columns of $W_q$ both span the same Krylov subspace $\mathcal{K}_q(A^{\mathrm{T}}, L^{\mathrm{T}})$, there exists a matrix $R_q$ such that

$$W_q = \widetilde{W}_q R_q \qquad (6)$$

$$R_q = \begin{bmatrix} R_{1,1} & R_{1,2} & \cdots & R_{1,q-1} & R_{1,q} \\ & R_{2,2} & \cdots & R_{2,q-1} & R_{2,q} \\ & & \ddots & \vdots & \vdots \\ & & & R_{q-1,q-1} & R_{q-1,q} \\ & & & & R_{q,q} \end{bmatrix}.$$

Matrix $R_q$ is upper block-diagonal; its block columns is generated iteratively. Let $\widetilde{R}_i$ $(i = 1, \cdots, q)$ be the block columns of the matrix $R_q$, i.e., $R_q = \begin{bmatrix} \widetilde{R}_1 & \cdots & \widetilde{R}_q \end{bmatrix}$. It can be shown that

$$\widetilde{R}_{j+1} = H_q \widetilde{R}_j, \qquad \text{for } j = 1, \cdots, q-1. \qquad (7)$$

Initially, $\widetilde{R}_1 = R_{11} E_1$, where $R_{11}$ is computed in the Block Arnoldi Algorithm and $E_1 = \begin{bmatrix} I_p & 0 & \cdots & 0 \end{bmatrix}^{\mathrm{T}}$. The matrix $H_q$ is computed by the Block Arnoldi Algorithm and takes the form of upper block Hessenberg matrix:

$$H_q = \begin{bmatrix} H_{1,1} & H_{1,2} & \cdots & \cdots & H_{1,q} \\ H_{2,1} & H_{2,2} & \cdots & \cdots & H_{2,q} \\ & H_{3,2} & \cdots & \cdots & H_{3,q} \\ & & \ddots & \vdots & \vdots \\ & & & H_{q,q-1} & H_{q,q} \end{bmatrix}$$

A derivation of the formulas above can be found in [8] for the SISO case. The matrix $M$ is solved from the following linear equation

$$R_q M^{\mathrm{T}} = -H_q \widetilde{R}_q. \qquad (8)$$

One should take the advantage of the upper block diagonality of $R_q$ when solving for $M$.

Finally, the matrix $B_q$ is not computed directly from the expression given in Step 3 of the OCFR algorithm. Rather, it is computed more efficiently by

$$B_q = W_q^{\mathrm{T}} B = R_q^{\mathrm{T}} \widetilde{W}_q^{\mathrm{T}} B. \qquad (9)$$

## 5. INTERPOLATION METHOD FOR PARAMETRIC MODELING

The compact reduced-order model resulting from the OCFR algorithm can be used for parametric interconnect modeling. In [5] an interpolation method was proposed for constructing parametric reduced order models. This technique is outlined briefly below. More details can be found in [5].

The PVL algorithm applied to a SISO model generates a reduced-order model with only $(3q-1)$ free parameters that determine the reduced-order model. The reduced-order model is represented by a $(3q-1)$-dimensional vector.

Suppose a full-order model involves a set of parameters. The interpolation idea is based on parameter value sampling. For the case of one parameter, Lagrange interpolation can be applied at the sample grids $p^{(i)}$, $i = 1, 2, \cdots, N$.

Let $\mathcal{M}^{(i)}$ be the $i$th model sampled at the parameter grid point $p^{(i)}$, for $i = 1, 2, \cdots, N$. Let $v^{(i)} \in \mathbb{R}^{3q-1}$ be the vector representing the $i$th reduced-order model computed by PVL from the $i$th full-order model $\mathcal{M}^{(i)}$. Reduced order models for parameter values other than the sampling points are obtained by interpolating the vectors $v^{(i)}$, $i = 1, 2, \cdots, N$.

The interpolation approach also can be applied to the reduced-order models generated by the OCFR algorithm. We have mentioned in Remark 1 that a reduced-order SISO model by the OCFR algorithm is determined by a $(2q)$-dimensional vector, which has less number of parameters than the models reduced by PVL.

## 6. EXPERIMENTS

We use three examples to demonstrate the application of the OCFR algorithm to multiple interconnect modeling and the combination of the OCFR algorithm with interpolation for parametric interconnect analysis.

The first example is used to demonstrate the application of OCFR algorithm to a MIMO model. The circuit consists of three coupled parallel wires, each wire is modeled by 9 RLC segments with coupling. For each segment, the lumped element values are $R = 5.3571m\Omega$, $L = 0.6138nH$, $C = 0.1420pF$, $CC = 0.0155pF$, $CL = 1pF$ where $CL$ is the load capacitance. We consider two voltage sources $V_{s1}$ and $V_{s2}$ as the inputs and measure the two voltage outputs at $V_1$ and $V_2$ (see Fig. 1).

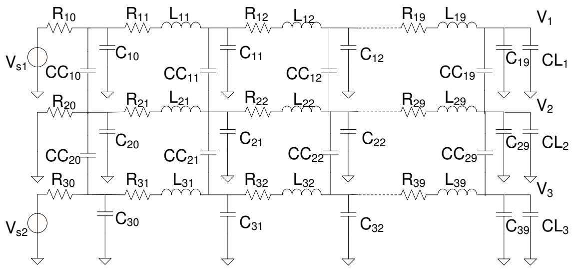

Fig. 1. Three coupled RLC lines.

The MNA formulation of the circuit leads to a full-order model of order 89. This model is reduced by the OCFR algotithm to the order order of 16. Shown in Fig. 2 are the frequency responses of $h_{12}(s)$ with the full-order model and the reduced-order model plotted together for comparison. We have checked that the reduced-order model has all poles in the left-half complex plane.

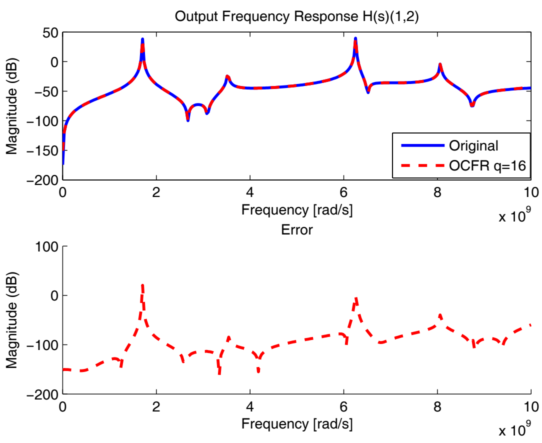


Fig. 2. Frequency responses of $h_{12}(s)$.

The coupling behavior of the multiple parallel interconnects can be studied by simulation in the time-domain.

Fig. 3 shows the the step responses of $h_{11}(s)$ and $h_{12}(s)$ compared to the Hspice transient simulation. By reducing to a model-order as low as $q = 9$, the reduced-order model still captures the timing behavior of the original interconnect system. Due to the low loss of the wires, the waves decay very slowly.

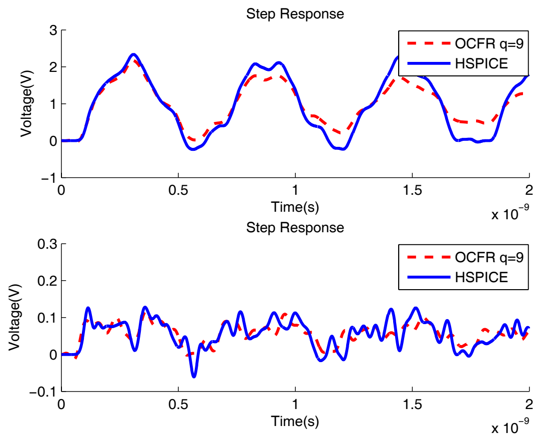


Fig. 3. Step responses of $h_{11}(s)$ and $h_{12}(s)$.

Application of OCFR to parametric modeling is demonstrated by the third example shown in Fig. 4. The nominal reference RLC values are assumed uniform with $R = 5.3571m\Omega$, $L = 0.6138nH$, and $C = 0.1420pF$. We choose $V_s$ as the input and $V_t$ as the output.

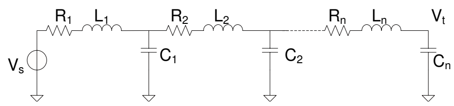

Fig. 4. An RLC line.

We would like to investigate the sensitivity of $L$ to the circuit. Assume the inductance $L$ varies in the interval $[0.5L_0, 1.5L_0]$, where $L_0$ is the nominal value. The interval is sampled by 20 equally spaced points and each of the sampled model is reduced by OCFR from 100th order to 15th order. Shown in Fig. 5 are the reduction results for two randomly chosen $L$ from the interval, computed by *OCFR + Interpolation*.

Next we consider that all the RLC parameters are allowed to vary in certain range around their nominal values. We choose $R \in [0.8R_0, 1.2R_0]$ with 6 grids, $L \in [0.8L_0, 1.2L_0]$ with 6 grids, and $C \in [0.8C_0, 1.2C_0]$ with 4 grids. Altogether, we need to collect 144 sample models and reduce them one-by-one by OCFR to the 8th

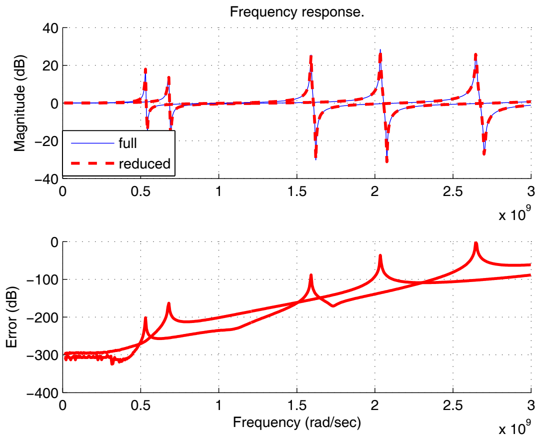


Fig. 5. Test results for single parameter perturbation (two cases).

order. Shown in Fig. 6 are the reduction results for two reduced-order models interpolated over the surrounding reduced sample models. The test results show that the interpolation technique has acceptable performance if used appropriately, which in certain applications can reduce computation drastically.

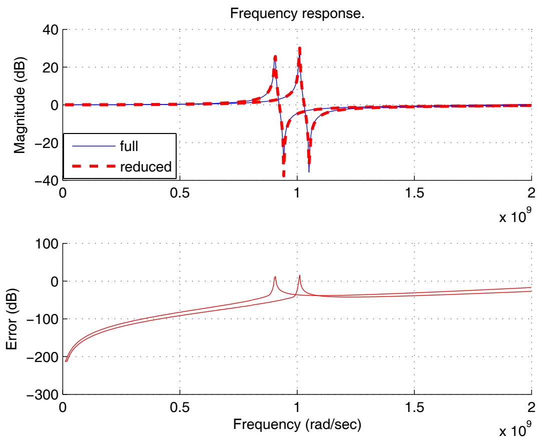


Fig. 6. Test results for multi parameter perturbation (two cases).

## 7. CONCLUSION

We have presented a new model order reduction technique using the observability canonical form in linear system theory. The reduced order models computed by this technique has the feature of small number of free parameters, thus are suitable for parametric reduced-order modeling. We have demonstrated using examples that the OCFR algorithm works reliably for MIMO interconnect modeling and in parametric analysis based on interpolation. This methodology will be improved in many aspects so that it can be applied to interconnect coupling analysis in the time-domain, statistical timing evaluation, signal integrity, and others.

## ACKNOWLEDGMENTS

This work was supported in part by the National Natural Science Foundation of China, Grant No. 60572028.

## REFERENCES

[1] S. R. Nassif, “Modeling and forecasting of manufacturing variations,” (embedded tutorial), *Proc. Asia South-Pacific Design Automation Conference (ASPDAC)*, pp. 145-150, Yokohama, Japan, January 2001.

[2] K. Agarwal, D. Sylvester, D. Blaauw, F. Liu, S. Nassif, and S. Vrudhula, “Variational delay metrics for interconnect timing analysis,” *Proc.Design Automation Conf.*, San Diego, CA, 2004, pp. 381-384.

[3] J.D. Ma and R. A. Rutenbar, “Fast interval-valued statistical modeling of interconnect and effective capacitance,” *IEEE Trans. on Computer-Aided Design*, vol. 25, no. 4, pp. 710–724, April 2006.

[4] G. Shi, B. Hu, and C.-J. R. Shi, “On symbolic model order reduction,” *IEEE Trans. on Computer-Aided Design*, vol. 25, no. 7, pp. 1257–1272, July, 2006.

[5] G. Shi and C.-J. R. Shi, “Parametri reduced order modeling for interconnect analysis,” Proc. Asia South-Pacific Design Automation Conference (ASP-DAC), Yokohama, Japan, 2004, pp. 774-779.

[6] T. Kailath, *Linear Systems*. Englewood Cliffs, New Jersey: Prentice Hall, Inc., 1980.

[7] P. Feldmann and R. Freund, “Efficient linear circuit analysis by Padé approximation via the Lanczos process,” *IEEE Trans. on Computer-Aided Design*, vol. 14, no. 5, pp. 639–649, May 1995.

[8] L. Deng. *Signal Integrity Analysis Based on Model Order Reduction*. Master Thesis, School of Microelectronics, Shanghai Jiao Tong University, 2007.

[9] Y. Saad. *Iterative Methods for Sparse Linear Systems*. PWS Publishing Company, Boston, USA, 1996.