Symbolic Time-Varying Root-Locus Analysis for Oscillator Design*

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***Abstract*—The small-signal analysis of an oscillator relative to a periodic steady-state (PSS) would generate periodic time-varying characteristic poles. Analyzing periodic root-loci can provide useful design information, which is not available from the existing circuit simulation tools. Although the numerical QZ algorithm can be used to generate periodic root-loci, this paper proposes an alternative symbolic computation method for repeated pole computation. It is demonstrated that the Muller algorithm can be used for finding the dominant periodic roots of a characteristic polynomial with periodic coefficients, whose efficiency is superior to the matrix-based numerical QZ method. Other advantages of symbolic root-locus analysis also are explored by applying the proposed method to the analysis of two oscillator circuits.**

***Index Terms*—Muller's Method, oscillator, periodic poles, symbolic method, time-varying root-locus (TVRL).**

I. Introduction

Oscillator is widely used in communication and data converter circuits. Its design relies on extensive use of simulation tools. The currently available commercial simulation tools provide very limited support for oscillator design exploration. Hence, developing new computer-aided design methods for oscillators is of practical importance.

An oscillator circuit can be described mathematically by the following nonlinear equation [1]

$$\frac{d}{dt}q(v(t)) + i(v(t)) + u(t) = 0, \quad v(0) = v_0, \tag{1}$$

where q, v, i, and u are n-dimensional vector functions with $u(t)$ denoting the excitation input, and v_0 the initial condition of the circuit. A self-oscillatory circuit has no driving input, i.e., $u(t) = 0$. Such a circuit would oscillate by a proper selection of an initial condition. For small-signal analysis, it is assumed that the vector functions $q(v)$ and $i(v)$ are differentiable with respect to v. Assume that the oscillator model (1) has a periodic steady-state (PSS) solution $v_{ss}(t) = v_{ss}(t+T)$ with a period T. Then $v_{ss}(t)$ satisfies

$$\frac{d}{dt}q(v_{ss}(t)) + i(v_{ss}(t)) = 0. \tag{2}$$

Assume that the steady-state trajectory is subject to small perturbation described by $v(t) = v_{ss}(t) + \delta v(t)$. Then we have

$$\frac{d}{dt}q(v_{ss}(t) + \delta v(t)) + i(v_{ss}(t) + \delta v(t)) = 0. \tag{3}$$

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Taking the first order Taylor expansions of $q(v)$ and $i(v)$ and using the steady-state equation (2), we obtain that

$$\frac{d}{dt}\left[C(t)\delta v(t)\right] + G(t)\delta v(t) = 0, \tag{4}$$

where

$$C(t) := \left.\frac{\partial q(v(t))}{\partial v}\right|_{v(t)=v_{ss}(t)}, \tag{5a}$$

$$G(t) := \left.\frac{\partial i(v(t))}{\partial v}\right|_{v(t)=v_{ss}(t)}. \tag{5b}$$

Ignoring the term $\left[\frac{d}{dt}C(t)\right]v(t)$, we get the following equation on the perturbation trajectory $\delta v(t)$,

$$C(t)\frac{d}{dt}\delta v(t) + G(t)\delta v(t) = 0, \quad \delta v(0) = \delta v_0, \tag{6}$$

where δv_0 is the initial perturbation. This is a linear time-varying differential equation describing the time-domain small-signal oscillator behavior. Since the coefficient matrices $C(t)$ and $G(t)$ in equation (6) are periodic functions, one would obtain a time-varying transfer function in the frequency-domain after taking a time-varying Laplace transform (see Zadeh [2]), which results in the following algebraic matrix equation

$$[sC(t) + G(t)]\delta V(s) = C(t)\delta v_0, \tag{7}$$

where $\delta V(s)$ is the Laplace transform of $\delta v(t)$. The poles of the small-signal model in the frequency-domain are the roots of s satisfying

$$D(s, t) := \det|sC(t) + G(t)| = 0, \tag{8}$$

where $D(s, t)$ denotes the time-varying characteristic polynomial of the oscillator. Since both $C(t)$ and $G(t)$ are periodic time-varying matrices, so are the roots of s. If we find the periodic roots by sampling the periodic steady-state orbit and plot the root-loci, we shall get periodic time-varying root-loci (TVRL) in the complex plane.

Shown in Fig. 1 is the Colpitts oscillator and its three root-loci, two of which are the pair of dominant (conjugate) roots that cross the $j\omega$ axis and one is the real root that moves along the negative real axis. The root-loci crossing the $j\omega$ axis are of physical implications which can be utilized to guide the oscillator optimization [3].

Finding the roots satisfying (8) can be treated in several ways. It can be solved as a generalized eigenvalue problem [4]. Typical numerical methods are the Modified-Decomposition

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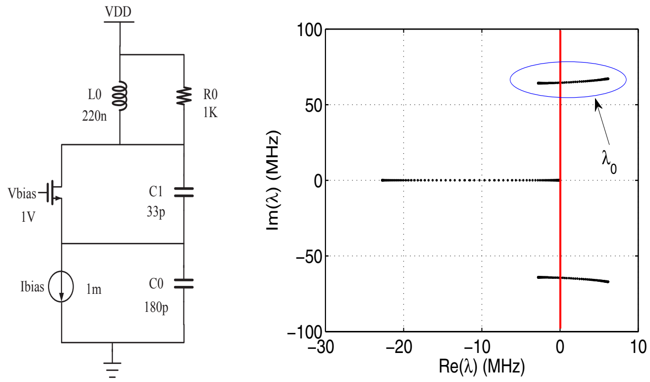
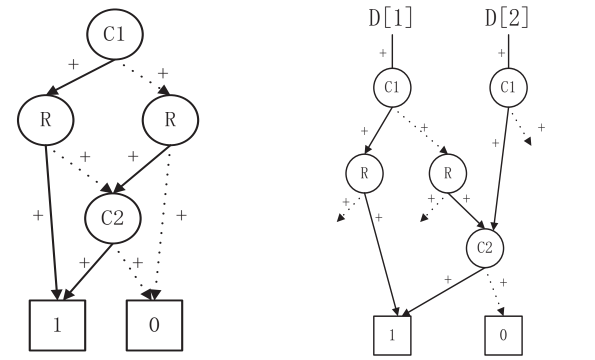


(a) (b)

Fig. 1. (a) Colpitts oscillator. (b) Three root-loci.



(a) (b)

Fig. 2. (a) A symbolic characteristic function in BDD. (b) The s-expanded multiple-root BDD, where $D[2]$ and $D[1]$ are respectively the coefficients of s^2 and s^1. The unterminated dotted arrows point to the terminal zero.

(MD) method or the QZ method. The QZ method was used by Broussev and Tchamov [3] to find the periodic root-loci of an oscillator. The root-finding problem also can be solved by constructing a symbolic polynomial $D(s,\, t)$ and then solve the dominant roots. This is the subject studied in this paper.

The symbolic TVRL computation method is introduced in section II. Experimental results are given in section III. Section IV concludes the paper.

II. Symbolic TVRL Analysis

The procedure of time-varying root-locus analysis is analogous to dc small-signal analysis, with the difference in whether the small-signal parameters are constant or time-varying.

A. Principle of symbolic analysis

Among many symbolic circuit analysis methods developed in the literature, we prefer to use a symbolic method based on binary decision diagram (BDD) [5]. A BDD-based symbolic method can construct exact symbolic expressions very fast for large analog circuits containing dozens of transistors. A symbolic BDD representation is also a computational data structure, on which one can implement a variety of algebraic operations. For example, it can be used to generate an s-expanded symbolic polynomial [6].

In this work, the time-varying polynomial coefficients are constructed by using a symbolic BDD. So far two BDD-based algorithms have been developed in the literature. The one by a graph reduction procedure can build a BDD whose vertices are directly the circuit small-signal parameters [7]. This form of symbolic BDD is much easier to use for optimizing circuit parameters [8].

For TVRL analysis, we are interested in the construction of a characteristic equation (8) for an oscillator, which is independent of the specification of input and output. By the graph-reduction procedure described in [7], a BDD as shown in Fig. 2(a) is constructed, which represents the characteristic polynomial

$$D(s) = (C_1 s)R^{-1} + (C_1 s)(C_2 s) + R^{-1}(C_2 s). \qquad (9)$$

The reader is referred to [7] for the details on how a BDD as given in Fig. 2(a) represents the symbolic terms. This polynomial can be rearranged in s-expanded form by direct manipulation on the BDD [6], resulting in a multi-root BDD shown in Fig. 2(b), each root representing one coefficient of s^k. The multi-root BDD are to be used for evaluating the time-varying coefficients of a characteristic polynomial.

B. Small-signal Model for TVRL Analysis

The commercial simulators like Synopsys HspiceRF [9] and Cadence SpectreRF [10] can be used for solving a numerical periodic steady-state (PSS) trajectory of an oscillator. Mayaram et al. in [11] presented an overview on the popular simulation algorithms and tools for radio-frequency (RF) circuits.

An appropriate small-signal model must be chosen for periodic small-signal analysis. This work uses the quasi-static small-signal MOSFET model shown in Fig. 3, which is compatible to the model used in the SpectreRF simulator.

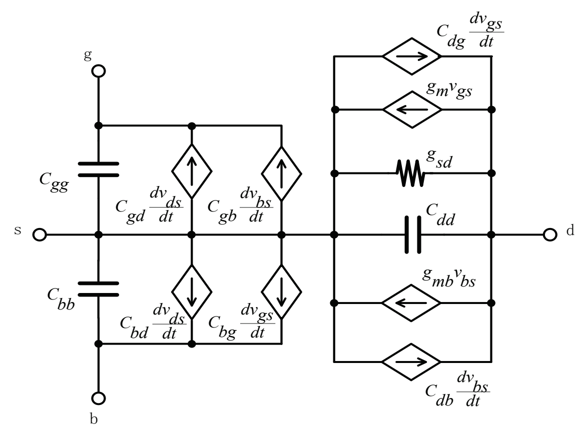


Fig. 3. Quasi-static small-signal MOSFET Model [12].

Six voltage-controlled capacitors are included in the small-signal model; they are treated in the symbolic tool as voltage-controlled current sources (VCCS) with transadmittances $C_k s$.

C. Polynomial root-finding

A traditional circuit simulator performs pole-zero analysis based on the modified nodal analysis (MNA) of a small-signal equivalent circuit, while a symbolic method in general performs pole-zero analysis by constructing polynomials. The Muller algorithm [13], [14] is a commonly used effective polynomial root-finding method. It is an iterative procedure in which three complex points are repeatedly updated until a single real root or a pair of conjugate roots are converged to. After polynomial deflation, the same search procedure is repeated. In implementation, circuit element scaling or frequency scaling can be applied for better convergence.

Three initial complex points must be specified to start the Muller algorithm. In TVRL analysis, a sequence of characteristic polynomials, denoted by $D(s, t_k) = 0$, are generated, where t_k is one time instant at which the periodic steady-state orbit is sampled. Since a sequence of points are sampled one after another along a periodic trajectory, the roots of the successively generated polynomials exhibit proximity in the complex plain. That is, the roots of $D(s, t_k) = 0$ would locate closely to the roots of $D(s, t_{k+1}) = 0$ if t_k and t_{k+1} are two successive sampling instants. In this sense, the last three points converged in the previous search for roots of $D(s, t_k) = 0$ can be used as the starting three points for the next Muller search of the roots of polynomial $D(s, t_{k+1}) = 0$. This strategy is called the "*successive Muller iteration*" for periodic polynomials, which is used for solving the dominant periodic root-loci. It helps accelerate the convergence of the Muller iteration.

In this work we only consider the dominant (conjugate) roots for TVRL analysis as proposed in [3]. Hence, the polynomial deflation and finding the roots of reduced order polynomials are not needed. Other non-dominant roots can be computed as well in principle, but their implications are not clear yet for the moment.

III. Experimental Results

The proposed symbolic TVRL analysis procedure was implemented in C++ and tested on a personal computer with an Intel Core2 2.26GHz processor and 256MB memory. The two oscillator circuits used in [3], shown in Fig. 4(a) and (b), were borrowed as the test circuits in this work. They are referred to as *Osc-1* and *Osc-2*, respectively. Both root-finding methods, the QZ algorithm and the *successive Muller iteration* algorithm, were implemented. The QZ algorithm is applied to the matrices $C(t)$ and $G(t)$ to solve all the roots of (8). We integrated the LAPACK QZ routine in our C++ program.

The PSS solutions were simulated with the Cadence SpectreRF using a TSMC 0.18um model library; the periodic node voltages at all time instants were saved in the files generated by PSS analysis [10]. Because the periodic small-signal model parameter values were not generated by the SpectreRF simulator, we managed to obtain such parameter values of all active

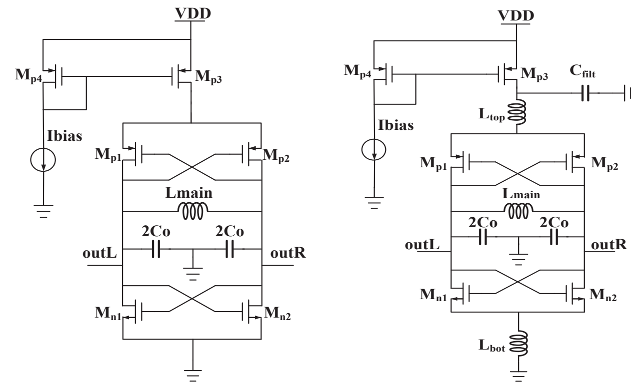


Fig. 4. Two oscillator circuits: (a) Cross-coupled oscillator (Osc-1), (b) Cross-coupled oscillator with bias inductors (Osc-2) [3].

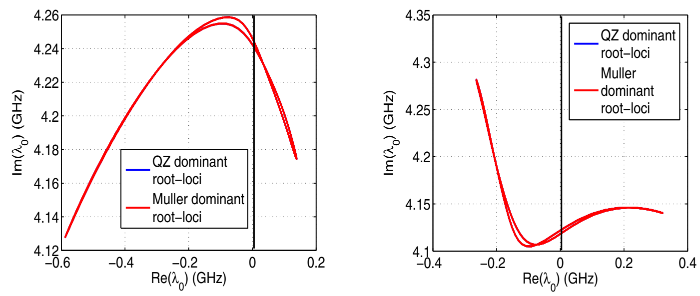


Fig. 5. Dominant root loci of circuit Osc-1 in (a) and Osc-2 in (b) calculated by the symbolic method and the QZ method.

TABLE I
PERFORMANCE COMPARISON BETWEEN THE SYMBOLIC METHOD AND THE QZ METHOD.

	Symbolic Method			QZ Method
	Coeff. eval. time (sec)	Muller dominant-root-finding time (sec)	Total time (sec)	Total root-finding time (sec)
Osc-1 (12 roots)	0.058	0.006	0.064	0.141
Osc-2 (16 roots)	0.215	0.009	0.224	0.260

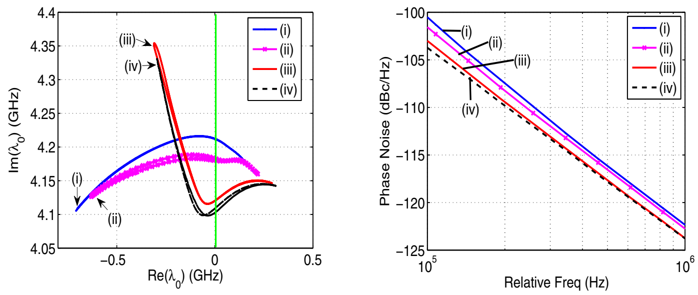
devices by running dc operating-point analysis with manually assigned the nodal voltages. Then the periodic small-signal device parameter values were passed to the symbolic program for generating the characteristic polynomials.

The quasi-static MOSFET small-signal model given in Fig. 3 was used in dc operating-point analysis and symbolic analysis. The inductors were modeled with a complete on-chip π-type model. The capacitors were assumed to have high Q and modeled ideally.

Shown in Fig. 5 are the dominant root-loci of Osc-1 and Osc-2 computed by the Muller method and the QZ method. The periodic steady-state orbits of both circuits were sampled by 202 points. The roots found by the two methods matched up to ten significant digits.

TABLE II
DETAILS WITH THE SYMBOLIC METHOD

	# symbols	s-expanded BDD size	BDD constr. time (inclu. s-exp)	s-coef. eval. time (s)
Osc-1	69	4,088	0.20	0.058
Osc-2	84	10,675	0.41	0.215



(a) (b)

Fig. 6. The effect of inductors on the oscillator performance: (a) root-loci for four combinations of L values: (i) $L_{top} = 0$ and $L_{bot} = 0$, (ii) $L_{top} = 0.8nH$ and $L_{bot} = 0$, (iii) $L_{top} = 0$ and $L_{bot} = 1.8nH$, and (iv) $L_{top} = 0.8nH$ and $L_{bot} = 1.8nH$; and (b) the corresponding phase noise curves.

We compared the computational efficiency between the numerical QZ method and the symbolic method. Shown in Table I are the time measures of the two methods applied to the two circuits Osc1 and Osc2. The runtime of the symbolic method was measured in several parts. The Muller's method was used to solve only the dominant roots. At the first time instant t_0, all roots of $D(s, t_0) = 0$ were calculated and the dominant complex one closest to the $j\omega$ axis and with positive imaginary part was chosen. Then the *successive Muller iteration* algorithm was applied to successively generate the dominant root-loci corresponding to the succeeding time instants. We see that most of the computation time was spent on calculating the polynomial coefficients, while the time for dominant root-finding was almost negligible.

For the symbolic method, the two circuits have respectively 69 and 84 symbols. The respective BDD construction time, BDD size, and the s-coefficient evaluation time are given in Table II. Since the symbolic BDD structure has quite large size, its numerical evaluation takes appreciable time. Several possibilities to improve the numerical evaluation time are by adopting a better symbol order [7] or by using currently popular GPU-based computation [15].

In contrast, the QZ method always solves all roots at every time instant from which the dominant roots are selected. The measured time includes the matrix build time and the LAPACK QZ solving time. The matrix sizes for the two oscillators are respectively 16×16 and 20×20.

From the design perspective, we know that the two inductors L_{top} and L_{bot} added in Osc-2 are intended for reducing the phase noise. With symbolic analysis, conceptually we can tune the circuit parameters to visualize the effect on the oscillator performance. To see this effect, we plotted in Fig. 6(a) four combination of the values of L_{top} and L_{bot}. The case (i) with $L_{top} = 0$ and $L_{bot} = 0$ corresponds to the topology of Osc-1. As the inductor values are tuned to appropriate values so that the LC resonance frequency is at the second harmonic of the oscillator, we see that the root-loci more closely embrace the $j\omega$ axis, which intuitively indicates that the oscillator phase noise is lowered. The plot in Fig. 6(b) verifies the claim; the phase noise is the lowest for the case (iv) when $L_{top} = 0.8nH$ and $L_{bot} = 1.8nH$.

IV. CONCLUSION

This paper has presented a fully symbolic approach to the time-varying root-locus analysis of an autonomous oscillator. Because the TVRL analysis requires repeated numerical computations of the characteristic roots, a symbolic method is suitable for this purpose because its data structure is constructed only once. Besides the computational efficiency, the symbolic method is advantageous to circuit topology or parameter optimization.

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